Int. J. Solids Structures, 1977, Vol. 13, pp. 1103–1122. Pergamon Press. Printed in Great Britain

THE EFFECT OF COUPLE STRESSES ON THE TIP OF A CRACK

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(Received 17 January 1977)

Abstract—The effect of couple stresses at a crack tip is investigated by considering two particular problems. A formally exact solution is obtained (for couple-stress and micropolar elasticity) for the case of a semi-infinite crack with a prescribed internal stress. Secondly, the problem of a finite crack in an infinite medium (with couple stresses) under uniform tension at infinity, is solved by matched expansions when the couple stress parameter is small compared with the crack length. In each case it is shown that the energy release rate from a crack tip tends to the classical elastic value as the couple stress (or micropolar) parameter tends to zero.

1. INTRODUCTION

The purpose of this paper is to clarify the effect of couple stresses on the stress concentration at the tip of a crack. This was first considered by Muki and Sternberg[8] who treated the problem of a finite length crack in an infinite medium under conditions of plane strain with a uniform tension acting at infinity. Their method consisted of reducing the problem to three Fredholm integral equations which were then solved numerically. The main conclusions were that, at the crack tips, the stress and couple stress fields had singularities of the same order, the order of the stress singularities being the same as those of the classical elastic problem. Graphical results were presented for the variation with l (the couple stress parameter) of the coefficients of the crack tip stress-singularities (stress intensity factors). It was found that the limit of the stress intensity factor as $l \to 0$ was different from that obtained with $l = 0$ for the usual solution without couple stresses. Another interesting result was that the ratio of the crack opening displacement at the centre for the medium with couple stresses to that obtained without couple stresses, tended to $1/(3-4\nu)$ as l/a tended to infinity, a being the crack length and ν Poisson's ratio.

Our contention is that the *energy release rate* (the important physical quantity from our point of view) does in fact tend to the classical elastic result when $l/a \to 0$, even though the stress intensity factor does not. In a previous note [1] the authors have demonstrated this for a model problem of a semi-infinite crack in a strip, for both the couple stress and micropolar elastic theories, difficulties in the analysis were circumvented by the use of a certain path independent integral.

Consideration is given here to two problems. (i) A semi-infinite crack in an infinite medium loaded by a specified internal stress, and (ii) a finite length crack in an infinite medium loaded by a uniform tension at infinity, this being the problem considered by Muki and Sternberg[8]. For problem (i) a solution is obtained by the Wiener–Hopf technique. The problem is uncoupled by solving successively a pair of Wiener–Hopf equations, and the results are valid for all values of the couple-stress parameter l. For problem (ii) a solution is obtained in the limit $l/a \to 0$ by the method of matched asymptotic expansions. As a check on the results obtained by matched expansions, and as an independent proof that the energy release rate tends to the classical elastic value as $l \to 0$, we give an argument involving an integral that is path-independent in the classical problem but is *not* in the couple-stress case. This integral, which is described in Appendix 1, can also be used together with the zero order inner solution of the asymptotic analysis to give the first order correction to the energy release rate.

The plan of the paper is as follows. In Section 2 we review the basic equations of couple-stress and micropolar elasticity as applied to plane strain conditions. Section 3 contains the analysis for the semi-infinite crack problem, firstly for the couple-stress case in some detail and then a brief description for the micropolar case. Numerical results are presented for the variation of the energy release rate with the couple stress (or micropolar) parameter. In Section 4 the finite crack is considered for the couple-stress case when l (the couple stress parameter is small. (We omit a treatment of the micropolar problem whose analysis would probably be very similar). Appendix 1 gives an alternative argument, using a certain integral, to show that the energy release rate tends to the classical result as the couple-stress (or micropolar) parameter tends to zero. Finally, Appendix 2 contains some of the details of the Wiener–Hopf factorisation procedure.

2. BASIC EQUATIONS

2.1 *Couple stress theory*

Since the problems to be considered are concerned with a state of plane strain we begin with the formulation given by Mindlin (1963) in terms of a generalisation of the Airy stress function. For the full equations and the background constitutive theory the reader is referred to the papers by Toupin[4] and Mindlin and Tiersten[7] or for the problem considered here to Muki and Sternberg[8]. The stress and couple stress are written

$$\left.\begin{aligned} \tau_{\alpha\beta} &= \epsilon_{\gamma\alpha}\epsilon_{\rho\beta}\phi_{,\gamma\rho} + \epsilon_{\gamma\alpha}\psi_{,\beta\gamma} \\ \sigma_\alpha &= \psi_{,\alpha} \\ \text{with} \qquad \epsilon_{12} = -\epsilon_{21} &= 1, \qquad \epsilon_{11} = \epsilon_{22} = 0, \end{aligned}\right\} \tag{2.1}$$

where ϕ and ψ are arbitrary (sufficiently smooth) stress functions, α and β taking the values 1 and 2. The components $\tau_{\alpha\beta}$ are the usual ones of the stress in the plane with cartesian coordinates (x_1, x_2) and the first index (α) denotes the direction of the outward normal. The couple stress σ_α $(\alpha = 1, 2)$ are abridged symbols for $\sigma_{\alpha 3}$, with all other stresses and couple stresses equal to zero, from the plane strain conditions, apart from τ_{33} and $\sigma_{3\alpha}$ which are given as

$$\tau_{33} = \nu\tau_{\alpha\alpha}, \qquad \sigma_{3\alpha} = 4\eta'\omega_{,\alpha}. \tag{2.2}$$

Here ν is Poisson's ratio, η' is an elastic constant arising in the couple stress theory, and ω is the component of the rotation vector in the x_3 direction perpendicular to the (x_1, x_2) plane. In terms of the stress functions ϕ and ψ the displacements and the rotation are given as

$$\begin{aligned} u_{(\alpha,\beta)} &= \frac{1}{2}\mu^{-1}[\epsilon_{\gamma\alpha}\epsilon_{\rho\beta}\phi_{,\gamma\rho} - \nu\delta_{\alpha\beta}\nabla^2\phi + \epsilon_{\gamma(\alpha}\psi_{,\beta)\gamma}] \\ 4\mu l^2\omega_{,\alpha} &= \psi_{,\alpha} \end{aligned} \tag{2.3}$$

where μ is the shear modulus and l the characteristic length parameter of the couple stress theory. In (2.3) $u_{(\alpha,\beta)}$ means $\frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha})$ and such a bracket in the subscripts will have a similar interpretation whenever it is used. The stress functions must satisfy the compatibility relations

$$(l^2\nabla^2\psi - \psi)_{,\alpha} = -2(1-\nu)l^2\epsilon_{\alpha\beta}\nabla^2\phi_{,\beta} \tag{2.4}$$

and

$$\nabla^4\phi = 0, \qquad l^2\nabla^4\psi - \nabla^2\psi = 0. \tag{2.5}$$

The problem then is to find ϕ and ψ and the corresponding stress and displacement field from the solution of eqns (2.5), and (2.4) and appropriate crack boundary conditions. In all the problems to be considered there is symmetry about $x_2 = 0$ so the problems can be reduced to half-plane problems for $x_2 \geq 0$ with boundary conditions on $x_2 = 0$ and suitable conditions on the stresses at infinity. We apply a Fourier transform over x_1 to eqns (2.4) and (2.5) and find, using the condition that $\bar{\phi}$ and $\bar{\psi}$ tend to zero as x_2 tends to infinity,

$$\begin{aligned} \bar{\phi} &= (C + Dx_2)\exp(-|s|x_2) \\ \bar{\psi} &= B\exp(-|s|x_2) + A\exp[-(s^2 + 1/l^2)^{1/2}x_2] \end{aligned} \tag{2.6}$$

where

$$B = -4(1-\nu)il^2sD,$$

in order to satisfy the compatibility conditions. In these formulae the bar denotes the Fourier transform, thus

$$\bar{\phi} = \int_{-\infty}^{\infty} \phi \, e^{isx_1} \, dx_1. \tag{2.7}$$

Similarly the following results are obtained for the Fourier transforms of the stresses, couple stresses and displacements

$$\left.\begin{aligned}
&\bar{\tau}_{21} = is\bar{\phi}_{,2} - s^2\bar{\psi}; \qquad \bar{\tau}_{12} = is\bar{\phi}_{,2} - \bar{\psi}_{,22}\\
&\bar{\tau}_{22} = -s^2\bar{\phi} - is\bar{\psi}_{,2}; \qquad \bar{\sigma}_1 = -is\bar{\psi}; \qquad \bar{\sigma}_2 = \bar{\psi}_{,2}\\
&-2\mu is\bar{u}_1 = (1-\nu)\bar{\phi}_{,22} + \nu s^2\bar{\phi} + is\bar{\psi}_{,2}\\
&-2\mu is\bar{u}_2 = (2-\nu)is\bar{\phi}_{,2} - s^2\bar{\psi} - \frac{(1-\nu)i}{s}\bar{\phi}_{,222}.
\end{aligned}\right\} \tag{2.8}$$

2.2 *Micropolar elasticity*

The main difference between the micropolar and couple stress theories is that the former theory gives extra degrees of freedom for local rotations, whereas in the couple stress theory the rotations are constrained; so that in our plane strain problem, for example, the rotation ω of eqn (2.3) is defined by $2\omega = u_{2,1} - u_{1,2}$. In the micropolar theory (see Eringen[2] for a fuller account) the stress equations of equilibrium are satisfied by the same stress functions given in (2.1). The couple stress σ_α would be called $m_{\alpha 3}$ in Eringen's notation. In place of the second equation of (2.3), we have

$$\gamma\phi_3 = \psi, \tag{2.9}$$

where ϕ_3 is the component of the micro-rotation vector in the x_3-direction and γ is a material constant. Whereas in the couple stress theory the strains e_{ij} are defined as in classical elasticity, in the micropolar theory we have as the only non-zero strains

$$\left.\begin{aligned}
e_{12} &= u_{2,1} - \phi_3\\
e_{21} &= u_{1,2} + \phi_3\\
e_{11} = u_{1,1}, &\qquad e_{22} = u_{2,2}
\end{aligned}\right\} \tag{2.10}$$

and the stress-strain relations

$$t_{kl} = \lambda e_{rr}\delta_{kl} + (\mu + \mathcal{K})e_{kl} + \mu e_{lk} \tag{2.11}$$

where λ, μ and $\mathcal{K}$ are material constants.

Equations (2.5) are still valid but eqns (2.4) are replaced by

$$(c_1^2\nabla^2\psi - \psi)_{,\alpha} = 2(1-\nu_1)b^2\epsilon_{\alpha\beta}\nabla^2\phi_{,\beta} \tag{2.12}$$

where

$$c_1^2 = \frac{\gamma(\mu+\mathcal{K})}{\mathcal{K}(2\mu+\mathcal{K})}, \qquad b^2 = \frac{\gamma}{2(2\mu+\mathcal{K})}, \qquad \nu_1 = \frac{\lambda}{(2\lambda+2\mu+\mathcal{K})}. \tag{2.13}$$

Finally the stress-equilibrium equations are

$$\left.\begin{aligned}
t_{lk,l} &= 0\\
\sigma_{lk,l} + \epsilon_{kmn}t_{mn} &= 0
\end{aligned}\right\} \tag{2.14}$$

where for plane strain $k = 3$ in $(2.14)_2$ and l and k take the values 1 and 2 in $(2.14)_1$. Equations (2.1), (2.5) and (2.12) are the solutions of these equations plus compatibility conditions.

Applying the Fourier transform to these equations gives, for the transformed stresses, the same as the first five equations of (2.8) with τ_{ij} replaced by t_{ij}. The transformed displacements are somewhat different, however, becoming

$$\left.\begin{aligned} -(2\mu+\mathcal{K})is\bar{u}_1 &= (1-\nu_1)\bar{t}_{11}-\nu_1\bar{t}_{22} \\ -(2\mu+\mathcal{K})is\bar{u}_2 &= (2\mu+\mathcal{K})\bar{\phi}_3+\frac{(\mu+\mathcal{K})}{\mathcal{K}}\bar{t}_{12}-\frac{\mu}{\mathcal{K}}\bar{t}_{21}. \end{aligned}\right\} \tag{2.15}$$

As in (2.6) we deduce for the Fourier transforms of ϕ and ψ,

$$\left.\begin{aligned} \bar{\phi} &= (C_1+D_1x_2)\exp(-|s|x_2) \\ \bar{\psi} &= B_1\exp(-|s|x_2)+A_1\exp(-\beta_1x_2) \end{aligned}\right\} \tag{2.16}$$

where

$$\beta_1=(s^2+1/c_1^2)^{1/2} \text{ and } B_1=-4(1-\nu_1)ib^2sD_1 \tag{2.17}$$

in order to satisfy the compatibility conditions.

3. SEMI-INFINITE CRACK PROBLEMS

3.1 *Couple stress theory*

As one of the simplest crack problems that is amenable to analytical treatment we consider a semi-infinite crack on $x_2=0$, $x_1<0$, with an internal stress $\tau_{22}=-\tau_0\exp(x_1/a)$ acting on the crack faces. For the couple stress theory this problem reduces to a half-plane problem with the following boundary conditions on $x_2=0$:—

$$\left.\begin{aligned} &\tau_{22}=-\tau_0 e^{x_1/a},\quad \sigma_2=0, &&\text{when}\quad x_1<0 \\ &\tau_{21}=0, &&\text{all}\quad x_1 \\ &u_2=0,\quad \omega_2=0, &&\text{when}\quad x_1>0. \end{aligned}\right\} \tag{3.1}$$

The corresponding conditions in the micropolar theory are given in Section 3.2. The Fourier transforms, with respect to x_1, of the boundary conditions (3.1) can be written

$$\left.\begin{aligned} &\bar{\tau}_{22}=-\tau_0a(1+isa)^{-1}+\bar{\tau}_+,\qquad \bar{\sigma}_2=\bar{\sigma}_+ \\ &\bar{\tau}_{21}=0,\quad \bar{u}_2=\bar{u}_-,\quad \bar{\omega}=\bar{\omega}_- \end{aligned}\right\} \tag{3.2}$$

where $\bar{\tau}_+$ and $\bar{\sigma}_+$ are the half-range transforms of τ_{22} and σ_2 for $x_2=0$, $x_1>0$. Thus, for example,

$$\bar{\tau}_+=\int_0^\infty \tau_{22}(x_1,0)\,e^{isx_1}\,dx_1. \tag{3.3}$$

Similarly $\bar{u}_-$ and $\bar{\omega}_-$ are half-range transforms of u_2 and ω for $x_1<0$, $x_2=0$, thus

$$\bar{u}_-=\int_{-\infty}^0 u_2(x_1,0)\,e^{isx_1}\,dx_1. \tag{3.4}$$

The subscripts + and − in these expressions denote functions which are regular in the respective upper and lower half planes of the complex s-plane.

The transformed potentials $\bar{\phi}$ and $\bar{\psi}$ have the forms (2.6) in terms of the four unknown function $A(s)$, $B(s)$, $C(s)$ and $D(s)$, with

$$B=-4(1-\nu)il^2sD. \tag{3.5}$$

At $x_2=0$, formulae (2.6) and (2.8) give the relations

$$\bar{\sigma}_+=|s|B-\beta A \tag{3.6}$$

$$4\mu l^2\bar{\omega}_- = A + B \tag{3.7}$$

$$\mu\bar{u}_- = (1-\nu)D \tag{3.8}$$

where

$$\beta^2 = s^2 + 1/l^2. \tag{3.9}$$

The boundary conditions (3.1) on τ_{21} and τ_{22} also show that

$$D - |s|C + is(B + A) = 0 \tag{3.10}$$

and

$$\bar{\tau}_{22} = -s^2C + is(|s|B + \beta A). \tag{3.11}$$

Here and henceforth all transforms are evaluated at $x_2 = 0$ unless explicitly stated otherwise, thus $\bar{\sigma}_+$ means $\bar{\sigma}_+(s, 0)$ for example.

Using the relations (3.5)–(3.11) the unknown functions A, B, C and D can be eliminated, and the problem reduced to equations which involve only unknown "plus" and "minus" functions, to be calculated by the Wiener–Hopf technique. For example, the following pair of relations can be deduced for C:

$$\left.\begin{aligned} -i|s|C &= 4\mu l^2 s\bar{\omega}_- - i\mu(1-\nu)^{-1}\bar{u}_- \\ \text{and}\qquad s^2C &= -is\bar{\sigma}_+ + \tau_0 a(1+isa)^{-1} - \bar{\tau}_+. \end{aligned}\right\} \tag{3.12}$$

On eliminating C between these equations we get

$$-\frac{i}{|s|}\{-is\bar{\sigma}_+ + \tau_0 a(1+isa)^{-1} - \bar{\tau}_+\} = 4\mu l^2 s\bar{\omega}_- - i\mu(1-\nu)^{-1}\bar{u}_-, \tag{3.13}$$

which can now be solved by a standard Wiener–Hopf procedure.

Firstly we require the factorisation of $|s|$ as a product of "plus" and "minus" functions of a complex variable s, and this is achieved by writing

$$|s| = s_+^{1/2}s_-^{1/2}, \tag{3.14}$$

where $s_-^{1/2}$ has a branch cut from 0 to $+i\infty$, and $s_+^{1/2}$ has a branch cut from 0 to $-i\infty$, both square root functions being positive when s is real and positive. It is convenient to regard $s_+^{1/2}$ and $s_-^{1/2}$ as the limit, $\epsilon \to +0$, of the respective functions $(s+i\epsilon)^{1/2}$ and $(s-i\epsilon)^{1/2}$ which have branch cuts from $\mp i\epsilon$ to $\mp i\infty$. Evidently (3.13) can be re-written as

$$\left.\begin{aligned} J(s) &= -s_+^{-1/2}(\bar{\tau}_+ + is\bar{\sigma}_+) + \tau_0 a(1+isa)^{-1}(s_+^{-1/2} - (i/a)_+^{-1/2}) \\ &= s_-^{1/2}(4\mu l^2 is\bar{\omega}_- + \mu(1-\nu)^{-1}\bar{u}_-) - (1+isa)^{-1}(i/a)_+^{-1/2}\tau_0 a. \end{aligned}\right\} \tag{3.15}$$

The first expression is regular in the upper half-plane and the second is regular in the lower half-plane, so that $J(s)$ defines a function regular in the whole plane by analytic continuation. It can be shown that J is bounded at infinity, hence by Liouville's theorem it is equal to a constant K which has yet to be determined. Equation (3.15) then gives the combinations $\bar{\tau}_+ + is\bar{\sigma}_+$ and $4\mu l^2 is\bar{\omega}_- + \mu(1-\nu)^{-1}\bar{u}_-$ in terms of K and τ_0. These expressions can then be used in (3.12) to determine C as

$$s^2C = Ks_+^{1/2} + \tau_0 s_+^{1/2}(i/a)_+^{-1/2}a(1+isa)^{-1}. \tag{3.16}$$

Knowing the function C, eqns (2.6), (3.2) and (3.5)–(3.11) can now be used to deduce a secondary Wiener–Hopf equation. For on eliminating all unknowns except $\bar{\sigma}_+$ and $\bar{u}_-$ we are led to the formula

$$-is_1\bar{\sigma}_+ = l^{1/2}K\beta_0 s_{1-}^{-1/2} + l^{1/2}\tau_0 a^{3/2}\beta_0 i_+^{-1/2}(1+is_1a_1)^{-1}s_{1-}^{-1/2} - \mu(1-\nu)^{-1}\beta_0 k\bar{u}_-. \tag{3.17}$$

where

$$k(s_1) \equiv 1 + 4(1-\nu)s_1^2(1 - |s_1|/\beta_0), \; \beta_0 = (s_1^2+1)^{1/2} \tag{3.18}$$

and

$$a_1 = a/l, \qquad s_1 = ls. \tag{3.19}$$

The expression β_0 has branch cuts from i to $i\infty$ and from $-i$ to $-i\infty$. In order to solve (3.17), the important first step is the factorisation of $\beta_0 k$ into the product of "plus" and "minus" functions. For β_0 this is done by inspection, and the factorisation for $k = k_+k_-$ is described in Appendix 2, using Cauchy's theorem. With this factorisation, (3.17) is arranged as

$$\frac{is_1\bar{\sigma}_+}{\beta_{0+}k_+} = \frac{\mu}{(1-\nu)}\beta_{0-}k_-\bar{u}_- - \frac{\beta_{0-}k_-s_{1-}^{-1/2}l^{1/2}}{k}\left\{K + \frac{\tau_0 a^{3/2}(i_+)^{-1/2}}{(1+is_1a_1)}\right\} \tag{3.20}$$

with

$$\beta_{0+} = (s_1+i)^{1/2}, \qquad \beta_{0-} = (s_1 - i)^{1/2}. \tag{3.21}$$

To solve (3.20) it is now necessary to decompose the right hand side into a sum of functions regular in the respective "plus" and "minus" regions. To achieve this, note the simple pole at $s_1 = i/a_1$ in the "plus" region and consider the function

$$s_1 m(s_1) \equiv \beta_{0-}k_-s_{1-}^{-1/2}/k \tag{3.22}$$

that occurs as a factor outside the curly bracket of (3.20). (The same function $m(s)$ occurs also in the finite crack problem of Section 4). In Appendix 2, $m(s_1)$ is written as a sum $(m_+ + m_-)$, and a few asymptotic properties are also derived there. Thus we write

$$s_1 m(s_1) = s_1(m_+ - c_0) + s_1(m_- + c_0) \tag{3.23}$$

with m_+, m_- and c_0 given explicitly, in integral form, in Appendix 2. Using (3.22) and (3.23) in (3.20), and subtracting out the pole at $s_1 = i/a_1$, we can rearrange (3.20) as

$$J_1(s_1) = \frac{is_1\bar{\sigma}_+}{\beta_{0+}k_+} + l^{1/2}Ks_1(m_+(s_1) - c_0) + \frac{l^{1/2}\tau_0 a^{3/2}(i_+)^{-1/2}}{1+is_1a_1}\{s_1(m_+(s_1)-c_0) - (i/a_1)(m_+(i/a_1) - c_0)\}$$
$$= \frac{\mu}{1-\nu}\beta_{0-}k_-\bar{u}_- - l^{1/2}Ks_1(m_-(s_1)+c_0) - \frac{\tau_0 l^{1/2}a^{3/2}(i_+)^{-1/2}}{1+is_1a_1}$$
$$\times\{s_1(m_-(s_1)+c_0) + (i/a_1)(m_+(i/a_1) - c_0)\}. \tag{3.24}$$

It is shown in Appendix 2 that, in the limit $|s| \to \infty$,

$$k_+ \sim 1, \quad k_- \sim 3 - 2\nu, \quad s_1(m_+ - c_0) \sim d, \quad s_1(m_- + c_0) \sim 1 - d + L/s_1, \tag{3.25}$$

where d and L are given explicitly in integral form. The edge conditions at the crack tip (i.e. that the stresses and couple stresses are singular like $r^{-1/2}$, where r is a small distance from the crack tip), imply that

$$\bar{\sigma}_+ = 0(s_+^{-1/2}) \quad \text{and} \quad \bar{u}_- = 0(s_-^{-3/2}) \quad \text{as} \quad |s| \to \infty. \tag{3.26}$$

These results and the generalised form of Liouville's theorem can be used to show that $J_1(s_1)$ is given by

$$J_1(s_1) = -Kl^{1/2}(1-d). \tag{3.27}$$

At this stage the constant K is determined from the condition that $\bar{\sigma}_+$ should not have a pole at $s = 0$ (i.e. the couple stress should tend to zero at infinity). This condition gives

$$K(1-d) = \tau_0 a^{1/2} l i_+^{1/2}(m_+(i/a_1) - c_0) \tag{3.28}$$

and the solution is now formally complete. Equations (3.24), (3.27) and (3.28) are sufficient to determine the stress and displacement fields. If we restrict attention to the neighbourhood of the crack tip we need only the behaviour of the transforms at large s. It is found that

$$\bar{\sigma}_+ \sim iKl^{1/2}s_{1+}^{-1/2} = iKs_+^{-1/2} \quad \text{as} \quad s \to \infty \tag{3.29}$$

and

$$\frac{3-2\nu}{1-\nu}\mu\bar{u}_- \sim \frac{i_+^{1/2}a^{1/2}\tau_0}{s_-^{3/2}}\left\{\frac{L(m_+(i/a_1)-c_0)}{1-d} - (1-d+i(l/a)(m_+(i/a_1)-c_0))\right\}. \tag{3.30}$$

From these two relations the couple stress and displacement near the crack tip can be determined. It is not difficult to show (c.f. Sih and Liebowitz[9]) that the fields near the tip can be written as

$$\left.\begin{aligned}
&\tau_{11} \sim -(1-2\nu)k_1(2r)^{-1/2}\left\{\cos\frac{1}{2}\theta - \frac{1}{2}\sin\theta\sin\frac{3}{2}\theta\right\}\\
&\tau_{22} \sim (3-2\nu)k_1(2r)^{-1/2}\left\{\cos\frac{1}{2}\theta - \frac{(1-2\nu)}{2(3-2\nu)}\sin\theta\sin\frac{3}{2}\theta\right\}\\
&\tau_{12} \sim -4(1-\nu)k_1(2r)^{-1/2}\left\{\sin\frac{1}{2}\theta + \frac{1-2\nu}{8(1-\nu)}\sin\theta\cos\frac{3}{2}\theta\right\}\\
&\tau_{21} \sim -\frac{1}{2}(1-2\nu)k_1(2r)^{-1/2}\sin\theta\cos\frac{3}{2}\theta\\
&\sigma_1 \sim -\frac{1}{2}ak_2(2r)^{-1/2}\sin\frac{1}{2}\theta, \qquad \sigma_2 \sim \frac{1}{2}ak_2(2r)^{-1/2}\cos\frac{1}{2}\theta\\
&\omega \sim \frac{1}{8}(a/\mu l^2)k_2(2r)^{1/2}\sin\frac{1}{2}\theta
\end{aligned}\right\} \tag{3.31}$$

The displacement field can be obtained from these formulae and the relations

$$2\mu u_{1,1} = (1-\nu)\tau_{11} - \nu\tau_{22}; \qquad 4\mu u_{2,1} = 4\mu\omega + \tau_{12} + \tau_{21}. \tag{3.32}$$

Thus from (3.29) and (3.30), k_1 and k_2 have the values

$$\left.\begin{aligned}
&k_1 = -\frac{\tau_0 a^{1/2}}{(3-2\nu)}\left\{\frac{L}{1-d}(m_+(i/a_1)-c_0) - (1-d) - \frac{il}{a}(m_+(i/a_1)-c_0)\right\}\\
&k_2 = -2la^{-1/2}i\tau_0(1-d)^{-1}(c_0 - m_+(i/a_1)).
\end{aligned}\right\} \tag{3.33}$$

As was indicated earlier, the important quantity from our point of view is the energy release rate G. This can be calculated by substituting the above expressions into the integral for G given in Appendix 1. The result is

$$G = \frac{1}{2}(\pi/\mu)\left\{(1-\nu)(3-2\nu)k_1^2 + \frac{1}{16}(a/l)^2k_2^2\right\}. \tag{3.34}$$

The expressions $m_+(i/a_1)$, c_0, d etc. are reduced to real integrals in Appendix 2. For tabulation purposes it is convenient to define $\bar{k}_1$, $\bar{k}_2$ and $\bar{G}$ by the relations

$$k_1 = \tau_0(3-2\nu)^{-1}a^{1/2}\bar{k}_1, \qquad k_2 = -2la^{-1/2}\tau_0\bar{k}_2 \tag{3.35}$$

and

$$G = \frac{1}{2}(a/\mu)\pi\tau_0^2(1-\nu)\bar{G}. \tag{3.36}$$

One object of the present analysis is to show that G tends to the classical elastic result as l

(the couple stress parameter) tends to zero. This is equivalent to showing that $\bar{G}$ tends to unity as l tends to zero. As can be seen from (3.33), (3.34) and the integrals in Appendix 2, the expression for $\bar{G}$ is complicated; apart from investigating it as a function of $(1-\nu)$ near ν equal to one it has not been verified analytically that $\bar{G}$ does tend to unity as $l \to 0$. However, for the finite length crack problem of Section 4 we have an independent argument (Appendix 1) to show that $\bar{G} \to 1$ as $l \to 0$, and numerical calculations for several values of ν demonstrate the result.

Table 1.

ν	l/a	$\bar{k}_2$	$\bar{k}_1$	$\bar{G}$
0.0	0	1.32	1.30	1.0
	0.01	1.28	1.30	0.98
	0.05	1.16	1.29	0.89
	0.10	1.04	1.27	0.81
	0.20	0.87	1.24	0.70
0.25	0	1.07	1.24	1.0
	0.01	1.04	1.24	0.98
	0.05	0.95	1.23	0.91
	0.10	0.86	1.22	0.84
	0.20	0.73	1.20	0.75
0.5	0	0.78	1.18	1.0
	0.01	0.76	1.18	0.98
	0.05	0.70	1.17	0.93
	0.1	0.64	1.16	0.88
	0.2	0.55	1.15	0.81

The values of $\bar{k}_1$ shown in Table 1 can be compared with results shown graphically by Muki and Sternberg[8, Fig. 4]. Our tabulated results are seen to be a little less than theirs; for example, when $\nu = 0.5$ and $l = 0$, their value of $\bar{k}_1 \approx 1.2$ is to be compared with our value of $\bar{k}_1 \approx 1.18$. Our contention is that our results are probably more accurate, since Muki and Sternberg need more computation to obtain numerical solutions of integral equations whose kernels are integrals involving Bessel functions. The solution of our problem, on the other hand, has been reduced to quadrature; and the striking result that $\bar{G} \to 1$ for $l \to 0$, with a variety of values for ν, seems confirmation of the accuracy of the result. We note further that $\bar{G}$ is a decreasing function of l, and is always less than unity.

3.2 *Micropolar theory*

The boundary conditions are the same as those of Section 3.1. Treating the problem as a half-plane problem we have on $x_2 = 0$:

$$\left.\begin{aligned} t_{22} = -\tau_0 e^{x_1/a}, \quad & m_2 = 0, \quad x_1 < 0 \\ t_{21} = 0, \quad & \text{all } x_1 \\ u_2 = 0, \ \phi_3 = 0, \quad & x_1 > 0 \end{aligned}\right\} \tag{3.37}$$

where m_i are abridged symbols for m_{i3} and we look for the solution in the half-plane $x_2 > 0$ with stress and couple stresses tending to zero at infinity.

This problem can be solved in an analogous way to that of Section 3.1, and the following results are obtained:- t_{11}, t_{22}, t_{12} and t_{21} have the same angular form as τ_{11} etc. of eqns (3.31) except that $(1-\nu)$ is replaced by $(1-\nu_1)b^2/c_1^2$ wherever $1-\nu$ occurs. The stress intensity factor k_1 then has a similar form to (3.33) and can be written as

$$k_1 = \frac{-\tau_0 a^{1/2}}{[1 + 2(1-\nu_1)b^2/c_1^2]} \left\{ \frac{L(m_+(i/a_1) - c_0)}{1-d} - (1-d) - \frac{ic_1}{a}(m_+(i/a_1) - c_0) \right\} \tag{3.38}$$

where $a_1 = a/c_1$. It should be noted that whereas the expressions in the curly brackets depended on $(1-\nu)$ in the expression for the kernel k of the couple-stress problem it is now replaced by $(1-\nu_1)b^2/c_1^2$ as a consequence of eqn (2.12). Having got the stresses the displacements can be evaluated via (2.15). For ease of reference we quote here results for normal stress and displacement near the crack tip

$$\left.\begin{aligned} t_{22} &\sim \{1+2(1-\nu_1)b^2/c_1^2\}k_1(2x_1)^{-1/2}, \quad x_2=0, \quad x_1>0 \\ u_2 &\sim \frac{2(1-\nu_1)}{(2\mu+\mathcal{K})}k_1(-2x_1)^{1/2}, \quad x_2=0, \quad x_1<0. \end{aligned}\right\} \tag{3.39}$$

The couple stress and displacement can also be evaluated via the method of Section 3.1 the result being

$$m_{23}=\frac{1}{2}ak_2(2r)^{-1/2}\cos\frac{1}{2}\theta, \qquad \phi_3=\frac{ak_2}{2\gamma}(2r)^{1/2}\sin\frac{1}{2}\theta \tag{3.40}$$

with

$$k_2=-\frac{2c_1}{a^{1/2}}\frac{i\tau_0}{(1-d)}(c_0-m_+(i/a_1)), \tag{3.41}$$

$a_1=a/c_1$ and c_1^2, b^2 and γ are related in eqns (2.13), and it should be noted again that where the argument $1-\nu$ occurred in c_0, d and m_+ in the couple stress problem it is now replaced by $(1-\nu_1)b^2/c_1^2$.

Using the above expressions the energy release rate G given by eqn (A1.1) of Appendix 1 can be evaluated. The result is

$$G=\frac{\pi(1-\nu_1)}{(2\mu+\mathcal{K})}\left\{[1+2(1-\nu_1)b^2/c_1^2]k_1^2+\frac{a^2k_2^2}{16(1-\nu_1)b^2}\right\}. \tag{3.42}$$

If we now write

$$k_1=\tau_0a^{1/2}\{1+2(1-\nu_1)b^2/c_1^2\}^{-1}\bar{k}_1, \qquad k_2=-2c_1a^{-1/2}\tau_0\bar{k}_2 \tag{3.43}$$

and

$$G=a\tau_0^2\pi(1-\nu_1)(2\mu+\mathcal{K})^{-1}\bar{G}, \tag{44}$$

then the results given in Table 1 apply where the argument $(1-\nu)$ is replaced by $(1-\nu_1)b^2/c_1^2$. Thus we see for example that $\bar{G}\to 1$ when $c_1\to 0$, whatever the value of $(1-\nu_1)b^2/c_1^2$ [note that $(1-\nu_1)b^2/c_1^2=\frac{1}{4}(1-\nu_1)(\mu+\mathcal{K})^{-1}\mathcal{K}$ is finite when $\gamma=0$].

In Table 2 we show some results which are more directly applicable to the micropolar case (they correspond to values of ν close to unity in the couple-stress case so are not physically relevant to the couple-stress case). These values are obtained by a numerical integration which could perhaps have been done more accurately, but the tabulated values are sufficient for our purpose. As indicated above, only the combination $(1-\nu_1)b^2/c_1^2$ is used; typical combinations that have been used by Eringen, for example, are $b/c_1=0.1$, $\nu_1=\frac{1}{2}$ whence $(1-\nu_1)b^2/c_1^2=0.005$; or $b/c_1=0.1$, $\nu_1=0$ giving $(1-\nu_1)b^2/c_1^2=0.01$. Both these cases are shown in Table 2 and the net effect of this choice of parameters is that their influence on the various stress intensity factors is small provided the choice of b/c_1 is so small that $(1-\nu_1)b^2/c_1^2$ is much less than unity. A similar effect is shown graphically in Eringen[2] for the circular hole.

Table 2.

$(1-\nu_1)b^2/c_1^2$	c_1/a	$\bar{k}_2$	$\bar{k}_1$	$\bar{G}$
0.005	0	0.009	1.002	1
	0.1	0.008	1.002	0.997
	0.5	0.005	1.0015	0.995
0.01	0	0.019	1.005	1
	0.1	0.016	1.004	0.995
	0.5	0.010	1.003	0.989

4. THE FINITE CRACK UNDER UNIFORM TENSION

Attention is given here to the problem of a finite traction-free crack in a transverse field of uniform tension, this being the problem treated by Sternberg and Muki who used an integral equation approach and found numerical solutions. The emphasis here is on the limiting form of the solution for small values of the couple stress parameter, making use of the method of matched asymptotic expansions.

Throughout most of the elastic space the limiting "outer" solution is obtained by simply setting $l = 0$ (hence $\sigma_\alpha = 0$) in the governing eqns (2.1)–(2.5) to get the classical solution with no couple stresses. This outer field is assumed to be valid at distances $r \gg l$, where r is the distance from either tip, and the solution will obviously depend on the precise geometry and boundary conditions at the crack.

It is found that the outer solution fails near the two tips where the effect of the couple stresses must be properly accounted for. The need for such "inner" solutions is indicated by a superficial glance at eqns (2.4), (2.5) where the terms in l involve high order derivatives; this is a common characteristic of singular perturbation problems and shows that the terms in l are potentially very large near the edges, no matter how small the value of l. Thus we should not neglect these terms within the small "inner" regions, near each tip, where r is of order l. But within these small regions it is reasonable to expect the field to be relatively insensitive to the gross features of the geometry. This suggests that the dominant length scale is l (rather than the crack width $2a$) and that the local inner problem, rescaled with respect to l, will be that of a semi-infinite crack whose solution is found by a Wiener–Hopf analysis.

Precise boundary conditions at infinity for these inner problems are to be determined by matching with the outer solution. For both inner and outer approximations are to hold in common regions

$$l \ll r \ll a$$

near each edge, and the two approximations must be asymptotically equivalent there. Details of the procedure are now given for the particular problem where a constant normal stress is prescribed on the crack.

Cartesian coordinates are chosen so that the crack is given by $x_2 = 0$, $|x_1| \leq a$, where τ_{22}, τ_{21} and σ_2 all vanish. At infinity we impose the condition

$$\tau_{22} \to \tau_0$$

with τ_{11}, τ_{12}, τ_{21} and σ_α vanishing at great distance from the crack.

Following Muki and Sternberg (1967), the problem is reduced to one of a crack with prescribed normal stress $(-\tau_0)$, by subtracting off the solution corresponding to undisturbed uniaxial tension. Thus we write

$$\left.\begin{aligned} u_1 = -\frac{1}{2}(\nu\tau_0/\mu)x_1 + u_1', \qquad u_2 = \frac{1}{2}(1-\nu)(\tau_0/\mu)x_2 + u_2' \\ \tau_{22} = \tau_0 + \tau_{22}', \quad \tau_{11} = \tau_{11}', \quad \tau_{12} = \tau_{12}', \quad \tau_{21} = \tau_{21}' \\ \omega = \omega', \quad \sigma_\alpha = \sigma_\alpha'. \end{aligned}\right\} \tag{4.1}$$

Thus the problem for τ_{ij}', u_α', ω', σ_α' is subject to the field eqns (2.1)–(2.5), with the boundary condition

$$\tau_{22}'(x_1, 0) = -\tau_0, \qquad |x_1| < a \tag{4.2}$$

with τ_{22}' vanishing at infinity.

The symmetry of the geometry and applied stress implies that we may confine attention to the half-space $x_2 \geq 0$. Symmetry considerations also lead to the following conditions on the plane $x_2 = 0$:

$$\left.\begin{aligned} \tau_{21}' = 0, \qquad \sigma_2' = 0, \qquad |x_1| < a \\ \tau_{21}' = 0, \qquad u_2' = 0, \qquad \omega' = 0 \qquad |x_1| > a. \end{aligned}\right\} \tag{4.3}$$

Thus the field eqns (2.1)–(2.5) have to be satisfied for $x_2 \geq 0$, subject to the boundary conditions (4.1)–(4.3), with τ'_{ij}, u'_i and σ'_{ij} vanishing at infinity, and edge conditions that limit the singularities in stresses and couple stresses to be of order $r^{-1/2}$ where r is the distance from either tip.

4.1 *Outer solution*

A first order solution, denoted by a superscript $^{(0)}$, is obtained by taking the classical limit $l = 0$ ($\sigma_{ij} = 0$). The solution to this classical problem is well-known and leads to the rotation and stress fields

$$\omega^{(0)}(x_1, 0) = -(\tau_0/\mu)(1-\nu)x_1(a^2 - x_1^2)^{-1/2}, \quad |x_1| < a \tag{4.4}$$

$$\tau_{22}^{(0)}(x_1, 0) = \tau_0\{|x_1|(x_1^2 - a^2)^{-1/2} - 1\}, \quad |x_1| > a \tag{4.5}$$

$$u_2^{(0)}(x_1, +0) = (\tau_0/\mu)(1-\nu)(a^2 - x_1^2)^{1/2}, \quad |x_1| < a. \tag{4.6}$$

Sternberg and Muki[8] point out that, for this class of plane geometries, the classical solution can readily be extended to give a solution of the *full* couple stress equations and boundary conditions, by including the couple stress field $\sigma_\alpha^{(0)} = 4\mu l^2 \omega_{,\alpha}^{(0)}$. In particular we have

$$\sigma_2^{(0)}(x_1, 0) = -4a^2 l^2 \tau_0(1-\nu)(x_1^2 - a^2)^{-3/2}, \quad |x_1| > a. \tag{4.7}$$

Although this supplemented outer solution satisfies all the governing equations, it is not acceptable as a uniformly valid solution of the given problem on account of the severe singularity of $\sigma_2^{(0)}$ at the edges. Thus we have to smooth out this singular behaviour by suitable inner solutions at the edges. Since $\sigma_2^{(0)}$ is of order l^2, it is negligibly small in the outer region and could be omitted from the leading outer approximation. It is retained for the moment, however, since it will prove useful in providing information on the expected behavior of our inner approximation for σ_2, which must match smoothly with the leading term $\sigma_2^{(0)}$.

In order to match near the edge $(a, 0)$ the limiting form of the outer solution is required at this point. Equations (4.4)–(4.6) have the limiting forms

$$\omega^{(0)}(x_1, 0) \sim -2^{-1/2}(\tau_0/\mu)(1-\nu)a^{1/2}(a - x_1)^{-1/2} \quad \text{as} \quad a - x_1 \to +0 \tag{4.8}$$

$$\tau_{22}^{(0)}(x_1, 0) \sim 2^{-1/2}\tau_0 a^{1/2}(x_1 - a)^{-1/2} \quad \text{as} \quad x_1 - a \to +0 \tag{4.9}$$

$$u_2^{(0)}(x_1, 0) \sim 2^{1/2}(\tau_0/\mu)(1-\nu)a^{1/2}(a - x_1)^{1/2} \quad \text{as} \quad a - x_1 \to +0. \tag{4.10}$$

and the supplementary function $\sigma_2^{(0)}$ of formula (4.7) is expected to have the behaviour

$$\sigma_2^{(0)} \sim -2^{1/2}a^{1/2}l^2\tau_0(1-\nu)(x_1 - a)^{-3/2} \quad \text{as} \quad x_1 - a \to +0. \tag{4.11}$$

This latter condition cannot be imposed upon our matched solution, for the function $\sigma_2^{(0)}$ is of order l^2 and is really to be neglected to leading order in the outer solution. The matched solution is uniquely determined by assuming only that σ_2 is of order l (or less). The self-consistency of the procedure is satisfactorily established by verifying finally that σ_2 is of order l^2 in the outer region and is indeed given by (4.11) near the edge. The magnitude of the supplementary function (4.7), and also the general form of the solution (4.28)–(4.42), leads us to anticipate that the next term in the outer solution will be of order l, but higher order terms are not found here.

4.2 *Inner solution*

It is obvious from the symmetry of our boundary value problem that τ_{22} and u_2 are even functions of x_1, whilst ω is odd with respect to x_1, and we may confine our attention to the tip at $(+a, 0)$. In the inner region near this edge the relevant length scale is l, so that new independent coordinates (X_1, X_2) are defined by the transformation

$$x_1 - a = lX_1, \qquad x_2 = lX_2. \tag{4.12}$$

The potentials ϕ' and ψ', and the displacement, stress and rotation tensors are rescaled according to the transformation

$$\left.\begin{aligned} \phi'(x_1, x_2) &= a^{1/2} l^{3/2} \Phi(X_1, X_2), & \psi'(x_1, x_2) &= a^{1/2} l^{3/2} \Psi(X_1, X_2) \\ \tau'_{ij}(x_1, x_2) &= (a/l)^{1/2} T_{ij}(X_1, X_2), & \sigma'_\alpha(x_1, x_2) &= (al)^{1/2} \Sigma_\alpha(X_1, X_2) \\ \omega'(x_1, x_2) &= (a/l)^{1/2} W(X_1, X_2), & u'_\alpha(x_1, x_2) &= (al)^{1/2} U_\alpha(X_1, X_2) \end{aligned}\right\} \tag{4.13}$$

where σ'_α $(\alpha = 1, 2)$ again means $\sigma'_{\alpha 3}$.

Under the transformation (4.13) the governing eqns (2.1)–(2.5) have (x_1, x_2) replaced by (X_1, X_2) and l replaced by unity. The other edge, at $X_1 = -2(a/l)$, disappears to $-\infty$ in the formal limit $l \to 0$. Thus in the first inner approximation we have to satisfy the modified eqns (2.1)–(2.5) together with the boundary conditions

$$\left.\begin{aligned} &U_2 = 0, \quad W = 0, \quad T_{21} = 0 \quad \text{for} \quad X_1 > 0, \quad X_2 = 0 \\ &T_{21} = 0, \quad \Sigma_2 = 0 \quad \text{for} \quad X_1 < 0, \quad X_2 = 0. \end{aligned}\right\} \tag{4.14}$$

Matching with the outer solution, through formulae (4.8)–(4.10) and the estimate $\sigma = 0(l)$, requires that

$$\left.\begin{aligned} &W(X_1, 0) \sim -2^{-1/2}(\tau_0/\mu)(1-\nu)(-X_1)^{-1/2} \quad \text{as} \quad X_1 \to -\infty \\ &T_{22}(X_1, 0) \sim 2^{\ 1/2}\tau_0 X_1^{\ 1/2} \quad \text{as} \quad X_1 \to \infty \\ &U_2(X_1, 0) \sim 2^{1/2}(\tau_0/\mu)(1-\nu)(-X_1)^{1/2} \quad \text{as} \quad X_1 \to -\infty \\ &\Sigma_2(X_1, 0) = 0(X_1^{\ 1/2}) \quad \text{as} \quad X_1 \to \infty. \end{aligned}\right\} \tag{4.15}$$

The expected behaviour (4.11) for σ_2 leads us to anticipate the more detailed behaviour

$$\Sigma_2(X_1, 0) \sim -2^{1/2}\tau_0(1-\nu)X_1^{-3/2} \quad \text{as} \quad X_1 \to \infty, \tag{4.16}$$

and this will be confirmed later. Satisfaction of the given boundary condition (4.1) is seen to imply that

$$T_{22}(X_1, 0) = 0 \quad \text{for} \quad X_1 < 0 \tag{4.17}$$

to this leading order, on account of the large factor $(a/l)^{1/2}$ that appears in (4.13) for the scaling of T_{22}. Evidently the first inner solution is an eigensolution of the semi-infinite problem and is scaled by the algebraic growth requirements (4.15) as $X_1 \to -\infty$. A solution by the Wiener–Hopf technique follows the procedure described in detail in Section 3. Thus Fourier transformations are defined as before, for example

$$\bar{\Phi}(s, X_2) = \int_{-\infty}^{\infty} \Phi(X_1, X_2)\, e^{isX_1}\, dX_1, \tag{4.18}$$

$$\bar{\Phi}_-(s, X_2) = \int_{\infty}^{0} \Phi\, e^{isX_1}\, dX_1, \qquad \bar{\Phi}_+(s, X_2) = \int_0^{\infty} \Phi\, e^{isX_1}\, dX_1. \tag{4.19}$$

The transforms of Ψ, U_2, W, Σ_{23} and T_{22} are denoted by $\bar{\Psi}$, $\bar{U}$, $\bar{W}$, $\bar{\Sigma}$ and $\bar{T}$.

The general form of the transform functions is very similar to that of Section 3, with l replaced by unity and $\bar{\tau}_{22}$, $\bar{\sigma}_2$, $\bar{\omega}$, $\bar{u}_2$, $\bar{\phi}$ and $\bar{\psi}$ replaced by $\bar{T}$, $\bar{\Sigma}$, $\bar{W}$, $\bar{U}$, $\bar{\Phi}$ and $\bar{\Psi}$. Thus the eqns (2.4) and (2.5) have the solutions

$$\bar{\Phi} = (C + DX_2)\, e^{-|s|X_2} \tag{4.20}$$

$$\bar{\Psi} = B\, e^{-|s|X_2} + A\, e^{\ \beta X_2} \tag{4.21}$$

with

$$\beta = (s^2 + 1)^{1/2} \tag{4.22}$$

and

$$B = -4(1-\nu)isD \tag{4.23}$$

are the forms corresponding to formulae (2.6). The functions A, B, C, D depend on s and are to be determined from the boundary conditions. The relations corresponding to (2.20) and (2.21) remain valid, subject to the forementioned notational changes.

From (4.16) we see that $\bar{T}_-(s, 0) = 0$, whence

$$\bar{T} = T_+ \tag{4.24}$$

where sufficies + and − again denote functions that are analytic in the upper and lower half s-plane respectively, and T_+ means $T_+(s, 0)$ for notational simplicity.

The matching conditions (4.15) are satisfied if the transforms have the behaviour

$$\left.\begin{aligned} W_- &\sim -\left(\frac{1}{2}\pi\right)^{1/2}\left(\frac{\tau_0}{\mu}\right)(1-\nu)\,e^{-(1/4)i\pi}s_-^{-1/2} \\ T_+ &\sim \left(\frac{1}{2}\pi\right)^{1/2}\tau_0\,e^{(1/4)i\pi}s_+^{-1/2} \\ U_- &\sim -\left(\frac{1}{2}\pi\right)^{1/2}(\tau_0/\mu)(1-\nu)\,e^{(1/4)i\pi}s_-^{-3/2}, \end{aligned}\right\} \quad \text{as} \quad s \to 0 \tag{4.25}$$

as can be seen by formally inverting each of these transforms to get the required forms (4.15); the difference between the functions W, T_{22}, U_2 and their limiting forms (4.15) have transforms that are continuous at $s = 0$. In the expressions (4.25) the functions $s_+^{1/2}$ and $s_-^{1/2}$ again have branch cuts from 0 to $\mp\infty$.

Since Σ_{23} is of order $X_1^{-1/2}$ as $X_1 \to \infty$, from formula (4.15), its transform is continuous at $s = 0$, and

$$\Sigma_+ = 0(1) \qquad \text{as} \qquad s \to 0. \tag{4.26}$$

The more precise information contained in formula (4.16) will be verified later; the simple bound (4.26) actually determines the solution.

On eliminating $A(s)$ and $C(s)$ from the equations (4.20), (4.21), using (2.20), (2.21), we are led to the primary Wiener-Hopf equation

$$T_+ + is\Sigma_+ = -|s|\left\{\frac{\mu}{1-\nu}U_- + 4i\mu sW_-\right\}, \tag{4.27}$$

which is similar to (3.23). Note that the minus function inside the curly brackets of formula (4.27) is closely related to the combination that emerges in the analysis of Sternberg and Muki[8, eqn 3.2].

On writing the kernel $|s|$ as a product $s_+^{1/2}s_-^{1/2}$ of "plus" and "minus" functions, formula (4.27) may be rearranged as

$$s_+^{-1/2}\{T_+ + is\Sigma_+\} = -s_-^{1/2}\left\{\frac{\mu}{1-\nu}U_- + 4i\mu sW_-\right\} \equiv F(s) \tag{4.28}$$

where the function F defined jointly by both sides of the equation is analytic except possibly at $s = 0$. Now for large s, each side of the eqn (4.28) is bounded, on account of the edge conditions (*c.f.* the argument following (3.25)). The requirements (4.25) and (4.26) at $s = 0$ imply the existence of a simple pole at the origin. Liouville's theorem then leads to the solution

$$F(s) = \left(\frac{1}{2}\pi\right)^{1/2}\tau_0\,e^{(1/4)i\pi}s^{-1} + F_0 \tag{4.29}$$

where F_0 is a constant to be determined, and is the value of $F(s)$ at infinity.

In particular the edge conditions give the asymptotic estimates

$$is\Sigma_+ \sim F_0 s_+^{1/2} \quad \text{and} \quad 4i\mu W \sim -F_0 s_-^{-3/2} \quad \text{as} \quad |s| \to \infty \tag{4.30}$$

in terms of the (unknown) constant F_0.

It still remains to calculate the individual terms T_-, Σ_+ and U_-, W_-; although the specifications (3.5)–(3.10) imply (4.28), the converse does not necessarily hold and we have to ensure that each of the individual modified identities (3.5)–(3.10) is satisfied. Thus on eliminating T_+ and U_- from these equations, using (4.28) and (4.29), we are led again to a secondary Wiener–Hopf equation

$$is\Sigma_+ = \beta_0 k \frac{\mu}{1-\nu} U_- + \beta_0 s_-^{-1/2} F(s) \tag{4.31}$$

for the functions Σ_+ and U_-, where the kernel $k(s)$ and the function are given by (3.18) with s in place of s_1. A connection with Sternberg and Muki's integral equation approach (1967) is again seen by comparing our kernel (3.18) with the integral eqn (3.8) of that work.

A formally exact solution of eqn (4.31) hinges on the factorisation

$$k = k_+(s)k_-(s) \tag{4.32}$$

and some properties of k_+ and k_- are given in Appendix 2, where it is shown that proportionality constants may be chosen so that

$$k_\pm(s) \sim 1 \quad \text{as} \quad |s| \to \infty. \tag{4.33}$$

Proceeding formally with our solution of (4.31), the equation can be rearranged to give

$$\frac{is\Sigma_+}{\beta_{0+}k_+} = \frac{\mu}{1-\nu}\beta_{0-}k_-U_- + \frac{k_-\beta_{0-}s_-^{-1/2}}{k}F$$

with β_{0+} and β_{0-} given by (3.21). The next step is to express the last term as a sum

$$k_-\beta_{0-}s_-^{-1/2}F/k \equiv N = N_+ + N_- \tag{4.34}$$

hence

$$\frac{is\Sigma_+}{\beta_{0+}k_+} - N_+ = \frac{\mu}{1-\nu}\beta_{0-}k_-U_- + N_- \tag{4.35}$$

is an analytic function of s, except possibly at $s = 0$.

The sum decomposition (4.34) is described briefly in the appendix. It is noted here that the sum is found, from (4.33) and (4.29), to have the form

$$N_+ + N_- \to F_0 \quad \text{as} \quad |s| \to \infty$$

while for small s we find

$$N_+ + N_- = 0(s_-^{-3/2}) + 0(1) \quad \text{as} \quad |s| \to 0.$$

Since we are free to add constants ($\pm N_0$) to N_+ and N_-, we may choose a decomposition such that

$$N_+ = 0 \quad \text{at} \quad s = 0 \tag{4.36}$$

with

$$N_- \to C_2 \quad \text{and} \quad N_+ \to F_0 \to C_2 \quad \text{as} \quad |s| \to \infty, \tag{4.37}$$

where C_2 is some constant that is to be found from the split (4.34) and constraint (4.36).

Now the function defined by (4.35) is readily seen from (4.25), (4.26) to have no singularity at $s=0$, and is seen from (4.30), (3.18) and (4.37) to have the limit C_2 as $|s|\to\infty$. Liouville's theorem ensures that this analytic function (4.35) is identically equal to C_2, and we have the solution

$$is\Sigma_+ = (N_+ + C_2\beta_{0+}k_+ \tag{4.38}$$

and

$$\frac{\mu}{1-\nu}U_- = (C_2 - N_-)/(\beta_{0-}k_-). \tag{4.39}$$

We are now able to complete the formal solution of the problem by evaluating the undetermined constant F_0 of (4.29). For in order that Σ_+ does not have a pole at $s=0$, in accordance with (4.26), the coefficient $(N_+ + C_2)$ of formula (4.38) must vanish at $s=0$. Hence from (4.36) we require

$$C_2 = 0, \tag{4.40}$$

and according to (4.37) the parameter F_0 must be chosen so that

$$N_- \to 0 \qquad \text{and} \qquad N_+ \to F_0 \qquad \text{as} \qquad |s|\to\infty. \tag{4.41}$$

It is shown in the Appendix 2 that this always possible if $(1-\nu)$ is positive, which is certainly the case since the Poisson ratio ν lies between -1 and $1/2$. Specifically it is found that

$$F_0 = \left(\frac{1}{2}\pi\right)^{1/2} \tau_0\, e^{(1/4)i\pi} \frac{c_0}{(1-d)} \tag{4.42}$$

where c_0 and d are defined by (A2.20) and (A2.24), and our formal solution is now complete.

4.3 *Solution near the tip*

The exact solution described above is now investigated at points very close to the tip, when $|X_1|\to 0$, and is determined from the behaviour of the transforms for large $|s|$. Thus from (4.38), with $C_2=0$, and using the asymptotics of Appendix 2 for N_+ and k_+, we see that

$$\Sigma_+ \sim s_+^{-1/2}\left(\frac{1}{2}\pi\right)^{1/2} \tau_0\, e^{-(1/4)i\pi} \frac{c_0}{(1-d)} \qquad \text{as} \qquad |s|\to\infty \tag{4.43}$$

hence

$$\Sigma_2 \sim -i2^{-1/2}\tau_0 X^{-1/2} c_0/(1-d) \qquad \text{as} \qquad X\to +0 \tag{4.44}$$

Similarly, it is found from the solution (4.39), with $C_2=0$, that

$$-\frac{\mu}{1-\nu}U_- \sim \frac{\tau_0\left(\frac{1}{2}\pi\right)^{1/2} e^{(1/4)i\pi}}{(3-2\nu)s_-^{3/2}}\left\{(1-d)+\frac{c_0 L}{(1+d)}\right\} \qquad \text{as} \qquad |s|\to\infty \tag{4.45}$$

where c_0, d and L are constants defined in the Appendix. It follows that the behaviour of U_2 near $X=-0$ is given by

$$\frac{\mu}{1-\nu}U_- \sim \frac{\tau_0 2^{1/2}(-X_1)^{1/2}}{(3-2\nu)}\left\{(1-d)+\frac{c_0 L}{1-d}\right\} \qquad \text{as} \qquad X_1\to -0. \tag{4.46}$$

Similar results can be obtained for W, as $X_1\to -0$, and for T_{22} as $X_1\to +0$. On rescaling these formulae in terms of the original variables, (4.12) and (4.13), it is seen that the formulae (4.44) and (4.46) are consistent with the general forms (3.37), with the multiplicative factors k_1 and k_2

now given by

$$k_1 = \frac{\tau_0 a^{1/2}}{3-2\nu}\left\{(1-d) + \frac{c_0 L}{(1-d)}\right\} \tag{4.47}$$

and

$$k_2 = -\frac{2ilc_0\tau_0}{a^{1/2}(1-d)} \tag{4.48}$$

in place of (3.33). Putting $l = 0$ in eqn (3.33), then $m_+(i/a_1) = 0$, and eqns (3.33) agree with (4.47) and (4.48). The numerical results given in Table 1 with $l = 0$ apply to the present problem.

4.4 *Behaviour of* Σ_2 *at large* X_1

The solution (4.38) for Σ_+, with $C_2 = 0$, can also be used to deduce the asymptotic limit

$$\Sigma_+ \sim \text{constant} + 2^{3/2}\pi^{1/2}\tau_0\, e^{-(1/4)i\pi}(1-\nu)s_+^{1/2} \quad \text{as} \quad s_+ \to 0.$$

It follows that, at large values of X_1,

$$\Sigma_2(X_1, 0) \sim -2^{1/2}\tau_0(1-\nu)X_1^{-3/2} \tag{4.49}$$

and this result confirms the predicted behaviour (4.16) that is necessary in order to match smoothly with the outer couple stress field (4.7), (4.11) that is described by Sternberg and Muki[8].

5. CONCLUDING REMARKS

For a crack in a tensile stress field, with either couple-stress or micropolar elastic theories, it has been shown that the energy release rate tends to the classical elastic result when the couple stress (or micropolar) parameter tends to zero. Also the energy release rate decreases as the couple stress parameter increases (Table 1). The result is consistent with the original idea of Mindlin, that the introduction of couple stresses would reduce the effect of holes as stress concentrators. Our opinion is that this energy release rate is the important physical quantity in the crack problem.

For the micropolar theory the same general trend is observed but as Table 2 shows, if b/c_1 is chosen to be much less than unity, the effect is hardly noticeable.

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APPENDIX 1. ENERGY RELEASE RATES FOR COUPLE STRESS AND MICROPOLAR ELASTIC MEDIA

1. *Couple stress theory*

For this theory we take as a starting point the definition

$$G = \int_c P_{j1} n_j ds \tag{A1.1}$$

where C is some contour enclosing the crack tip (or the right hand tip in the case of the finite crack) and n_j is the outward normal from C. The tensor P_{jt} is defined as

$$P_{jt} = W\delta_{jt} - \tau_{ji}u_{i,t} - \sigma_{ji}\omega_{i,t} \tag{A1.2}$$

where $\mathscr{W}$ is the strain energy function; for the plane–strain problems considered here $\mathscr{W}$ is given as

$$4\mu\mathscr{W} = \tau_{(\alpha\beta)}\tau_{(\alpha\beta)} - \nu\tau_{\alpha\alpha}\tau_{\beta\beta} + \frac{1}{2l^2}\sigma_{\alpha 3}\sigma_{\alpha 3} \quad \text{(A1.3)}$$

with α and β equal to 1 or 2. The summation convention is used and as before $\tau_{(\alpha\beta)}$ means $\frac{1}{2}(\tau_{\alpha\beta} + \tau_{\beta\alpha})$. It is asserted that the expression (A1.1) can be derived from the energy balance equation and is interpreted as the energy release rate for crack growth, but the derivation is omitted here.

Using (A1.2) and (A1.3) it can be shown by direct, though tedious, calculation that

$$P_{jt,j} = 0 \quad \text{(A1.4)}$$

In particular, this shows that the integral (A1.1) is path independent by a simple application of the divergence theorem.

In the classical elastic case other invariant integrals besides G have been derived by Günther[4] (see also Eshelby[3] and Knowles and Sternberg[5]).

One of these is the integral M, defined as

$$M = \int_c x_t P_{jt} n_j ds, \qquad (t, j = 1 \text{ or } 2). \quad \text{(A1.5)}$$

In the couple stress case this integral is not path independent, for a direct calculation shows that

$$(x_t P_{jt})_{,j} = 4\mu l^2 \omega\omega_{,jj} \quad \text{(A1.6)}$$

where it is recalled that

and

$$\left.\begin{aligned} -4\mu l^2 \omega_{,jj} &= \tau_{12} - \tau_{21} \\ \omega &= \frac{1}{2}(u_{2,1} - u_{1,2}). \end{aligned}\right\} \quad \text{(A1.7)}$$

The M-integral (A1.5) can now be used to confirm, for the finite crack problem of Section 4, that as $l \to 0$ the energy release rate G tends to the classical value $G^{(0)}$ without couple stresses. To demonstrate this result consider the integral M evaluated round a closed contour C_1 (Fig. 1) that just surrounds the crack with circular loops of small radius enclosing the ends.

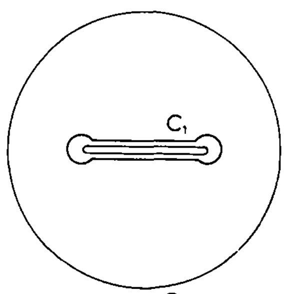

Fig. 1.

There is no contribution from the straight line sections of C_1, since the integrand $x_1 P_{21}$ vanishes there on account of the stress-free boundary conditions (τ_{22}, τ_{21} and σ_2 equal to zero). The symmetry of the problem ensures that the circular paths contribute equally, and since $x_1 \to \pm a$ on these arcs when $\epsilon \to 0$, we have

$$M = 2a \int P_{j1} n_j ds = 2aG \quad \text{(A1.8)}$$

using the definition (A1.1).

Now the use of the divergence theorem with the function $x_j P_{jt}$ within the region S, bounded by a large circle C and outside C_1, shows from (A1.6) and (A1.8) that

$$2aG = \int_{C_\infty} x_j P_{jt} n_t ds + 4\mu l^2 \int_S \omega\omega_{,jj} dS.$$

A similar calculation applied to the classical solution (with no couple stresses) shows that

$$2aG^{(0)} = \int_{C_\infty} x_j P_{jt}^{(0)} n_t ds,$$

where the superscript $^{(0)}$ again refers to the classical problem, and the area integral is absent since $\omega_{,jj}^{(0)} = 0$ in the classical problem.

Now the asymptotic solution described in Section 4 shows the outer solution of the couple stress problem to be asymptotically equal to the classical solution, with an anticipated error of order l in the outer region. Since C_∞ is in this outer region, we have $P_{jt} = P_{jt}^{(0)} + O(l^2)$, and on subtracting the equations above, we find

$$2a(G - G^{(0)}) = 4\mu l^2 \int_S \omega\omega_{,jj} dS + O(l) \tag{A1.9}$$

as $l \to 0$, where S now denotes the whole space outside the crack.

It remains only to show that the right hand side of (A1.9) vanishes as $l \to 0$. Now $\omega_{,jj}$ vanishes in the outer region, so we have to integrate over the regions near to the two edges, which give equal contributions by symmetry. Now in view of the scalings (4.12), (4.13) we find that

$$G - G^{(0)} \sim 4\mu l \int_{-\pi}^{\pi} \int_0^{\infty} WW_{,jj} R dR d\Theta + O(l) \tag{A1.10}$$
$$= O(l)$$

where (R, Θ) are polar coordinates referred to the inner coordinates (X_1, X_2).

An evaluation of the error term (A1.10) requires an exact evaluation of the integral $\int (\nabla W)^2 dS$, and some information about the next order outer solution, so does not seem to be amenable to analytical treatment. Nevertheless, formula (A1.10) does provide some useful information about G. For it verifies that G does indeed tend to $G^{(0)}$ as $l \to 0$, the error term being of order l. The convergence of the integral (A1.10), is readily confirmed from the known limiting behaviour of W, given in Section 4, for R large and small.

2. *Micropolar Theory*

This case is quite similar to the couple stress theory, the energy release rate G being defined as an integral of the form (A1.1), with

$$P_{jt} = \mathcal{W}\delta_{jt} - t_{ji}u_{i,t} - m_{ji}\phi_{i,t} \tag{A1.11}$$

in place of (A1.2). Here $\mathcal{W}$ is the strain energy function for a linear micropolar solid in plane strain and is given by

$$2\mathcal{W} = \lambda e_{kk}e_{ll} + (\mu + \mathcal{K})e_{kl}e_{kl} + \mu e_{kl}e_{lk} + \gamma(\phi_{3,1}^2 + \phi_{3,2}^2), \tag{A1.12}$$

where l, $k = 1, 2$ and so $\delta_{ll} = 2$.

Direct calculation again shows that

$$P_{jt,j} = 0 \tag{A1.13}$$

and

$$(x_t P_{jt})_{,j} = \mathcal{K}\phi_3(e_{21} - e_{12}) = \gamma\phi_3\phi_{3,jj} \tag{A1.14}$$

A consequence of formulae (A1.13) and (A1.14) is that the integral G, defined as eqn (A1.1), is again path independent, but the M integral (A1.5) is not. These integrals can be used to show that G tends to the classical elastic result, as γ tends to zero, in an analogous fashion to that of the couple stress case.

APPENDIX 2 WIENER-HOPF FACTORISATIONS

In order to factorise the kernel $k(s)$, of formulae (3.18) and (4.31), into a product $k_+(s)k_-(s)$, it is convenient to consider the function to be the limit, as $\epsilon \to 0$, of the expression

$$k(s;\epsilon) = 1 + 4s^2(1-\nu)\{1 - (s^2+\epsilon^2)^{1/2}(s^2+1)^{-1/2}\} \tag{A2.1}$$

in which $(s^2+\epsilon^2)^{1/2}(s^2+1)^{-1/2}$ is real and positive when s is real and positive, with branch cuts from $i\epsilon$ to i and from $-i\epsilon$ to $-i$. At large s,

$$k(s;\epsilon) \sim 1 + 2(1-\nu)(1-\epsilon^2) \equiv k_0 \tag{A2.2}$$

with

$$k_0 = 3 - 2\nu \quad \text{when} \quad \epsilon \to 0. \tag{A2.3}$$

and k has no zeros in its cut plane. On writing

$$\log(k/k_0) = \log k_+ + \log(k_-/k_0),$$

the usual Cauchy integral representations for $k_\pm$ are given by

$$\left.\begin{matrix}\log k_+(s) \\ \log(k_-(s)/k_0)\end{matrix}\right\} = \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log\{k(z)/k_0\}}{z-s} dz \tag{A2.4}$$

with $im\, s > 0$ for k_+ and $im\, s < 0$ for k_-, and the constant k_0 has been included with k_- in order to ensure that

$$k_+ \to 1 \quad \text{as} \quad |s| \to \infty. \tag{A2.5}$$

It follows that

$$k_- \to k_0 \qquad \text{as} \qquad |s| \to \infty, \tag{A2.6}$$

with k_0 given by (A2.3) when ϵ has the limiting value zero. The integral expressions have indented contours of integration when $im\ s \to 0$.

Turning our attention to the integral for $\log\{k_-(s)/k_0\}$, the integration path may be collapsed around the upper branch cut to get

$$\log\{k_-(s)/k_0\} = -\frac{1}{\pi}\int_0^1 \tan^{-1}\left\{\frac{4y^3(1-\nu)}{(1-y^2)^{1/2}[1-4y^2(1-\nu)]}\right\}\frac{dy}{y+is} \tag{A2.7}$$

on letting $\epsilon \to 0$. Although defined initially in the lower half s-plane, this formula provides a continuation that defines a function except for a cut from $s = 0$ to $s = i$.

On setting $s = -it$, formula (A2.7) also shows that $k_-(-it)/k_0$ is real and positive when t is real and positive.

Since the integration range is finite in (A2.7), the integrand may be formally expanded for large s to get a series in inverse powers of s. Thus we find

$$k_-(s)/k_0 \sim 1 + \frac{i}{\pi s}\int_0^1 \tan^{-1}\left\{\frac{4y^3(1-\nu)}{(1-y^2)^{1/2}[1-4y^2(1-\nu)]}\right\}dy; \tag{A2.8}$$

since $k(s) = k_0 + 0(s^{-2})$ at large s, we obviously have

$$k_+(s) \sim 1 - \frac{i}{\pi s}\int_0^1 \tan^{-1}\left\{\frac{4y^3(1-\nu)}{(1-y^2)^{1/2}[1-4y^2(1-\nu)]}\right\}dy. \tag{A2.9}$$

The secondary Wiener–Hopf problems of Sections 3 and 4 require sum decompositions for functions involving k_- and k_+. Both cases can conveniently be handled in terms of an auxiliary function $m(s)$ that is now defined as

$$m(s) = \beta_{0-}(s)s_-^{-3/2}/k_+(s) = k_-(s)\beta_{0-}(s)s_-^{-3/2}/k(s) \tag{A2.10}$$

with

$$\beta_{0-}(s) = (s-i)^{1/2}. \tag{A2.11}$$

The function $m(s)$ has branch cuts from 0 to $i\infty$ and from $-i\epsilon$ to $-i$, and we now seek its sum split of the form

$$m(s) = m_+(s) + m_-(s). \tag{A2.12}$$

It is readily found that

$$m(s) \sim 1/s \qquad \text{as} \qquad |s| \to \infty, \tag{A2.13}$$

while for small s we have

$$m(s) = k_-(s-i)^{1/2}s_-^{-3/2}\{1 - 4(1-\nu)s^2 + 0(s^2|s|)\} \qquad \text{as} \qquad s \to 0, \tag{A2.14}$$

where the first pair of terms are "minus" functions. Thus we are led to anticipate the results

$$m_+ = 0(1) \qquad \text{and} \qquad m_- = 0(s^{-3/2}) \qquad \text{as} \qquad s \to 0. \tag{A2.15}$$

Since equal and opposite constants can be added to m_+ and m_- we can expect to normalise by the choice

$$m_+ = 0 \qquad \text{when} \qquad s = 0. \tag{A2.16}$$

The Cauchy integral identities

$$m_\pm(s) = \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{m(z)\,dz}{z-s} \pm c_0, \tag{A2.17}$$

with c_0 to be found, provide a starting point for our investigation of m_+ and m_-. On collapsing the integration paths round the *lower* branch cut, and thereby picking up a residue contribution if $im\ s < 0$, we find

$$m_+(s) = -\frac{1}{2\pi i}\int_\Gamma \frac{m(z)dz}{z-s} + c_0 \tag{A2.18}$$

and

$$m_-(s) = \frac{1}{2\pi i}\int_\Gamma \frac{m(z)dz}{z-s} + m(s) - c_0 \tag{A2.19}$$

where Γ is the anti-clockwise loop enclosing the branch cut from $-i\epsilon$ to $-i$. There is no contribution from the large semi-circle at infinity, on account of the decay (A2.13) in m. In order to satisfy (A2.16) we choose c_0 to be

$$c_0 = \frac{1}{2\pi i}\int_{\Gamma} \frac{m(z)}{z}\,dz = -\frac{1}{2\pi i}\int_0^1 [m(-iy)]\frac{dy}{y} \tag{A2.20}$$

where $[m(-iy)] = m(-iy+0) - m(-iy-0)$, is the discontinuity in m across the lower cut. Using formula (A2.10) and noting that $k_-\beta_{0-}s_-^{3/2}$ is continuous across the lower cut, the discontinuity in m is obviously related to the jump in $(1/k(s))$, which is readily calculated from the definition (A2.1) with ϵ then set equal to zero. Thus we find

$$[m(-iy)] = \frac{-8y^{3/2}(1-\nu)(1-y^2)^{1/2}(1+y)^{1/2}k_-(-iy)}{1-y^2(9-8\nu)+8y^4(1-\nu)(3-2\nu)}. \tag{A2.21}$$

To investigate the behaviour of $m_\pm$ at large s, we may simply expand the integrands in formulae (A2.18), (A2.19) in inverse powers of s. This gives the results

$$m_- \sim -c_0 + (1-d)s^{-1} + Ls^{-2} \tag{A2.22}$$

and

$$m_+ \sim c_0 + ds^{-1} \quad \text{as} \quad |s| \to \infty, \tag{A2.23}$$

with c_0 given by (A2.20), and

$$d = \frac{1}{2\pi}\int_0^1 [m(-iy)]dy, \tag{A2.24}$$

$$L = \frac{i}{\pi}\int_0^1 \tan^{-1}\left\{\frac{4y^3(1-\nu)}{(1-y^2)^{1/2}[1-4y^2(1-\nu)]}\right\}dy - \frac{1}{2}i + \frac{i}{2\pi}\int_0^1 y[m(-iy)]dy. \tag{A2.25}$$

The sum decomposition can now be determined for the function $N(s)$ that occurs in Section 4. By definition

$$N = F_1 m(s) + F_0 s m(s) \tag{A2.26}$$

where

$$F_1 = \tau_0\left(\frac{1}{2}\pi\right)^{1/2} \exp\left(\frac{1}{4}\pi i\right) \tag{A2.27}$$

and F_0 is the (unknown) constant of formula (4.29). Define the functions

$$N_+ = F_1 m_+ + F_0 s(m_+ - c_0) \tag{A2.28}$$

$$N_- = F_1 m_- + F_0 s(m_- + c_0), \tag{A2.29}$$

which have the required sum (A2.26) and which are chosen so that

$$N_+ = 0 \quad \text{when} \quad s = 0.$$

The terms involving c_0 have been chosen to ensure that $N_\pm$ are bounded at infinity. Specifically,

$$\left.\begin{aligned} &N_- \to -F_1 c_0 + F_0(1-d) \\ \text{and} \\ &N_+ \to F_1 c_0 + F_0 d \end{aligned}\right\} \tag{A2.30}$$

at large s, with c_0 and d given by (A2.20) and (A2.24). The analysis of Section 4 requires F_0 to be chosen so that N vanishes at infinity, so that

$$F_0 = c_0 F_1/(1-d). \tag{A2.31}$$

One final detail is that of proving $d \neq 1$, which is now established by showing that d is real and negative. For the denominator of (A2.21) is positive in the relevant region $0 \leq y \leq 1$; according to (A2.7) and (A2.3) the function $k_-(-it)$ is real and positive when $3-2\nu > 0$. Thus from (A2.24) and (A2.21) it is seen that d is certainly negative when $\nu < 1$, and this is certainly the case for all possible values $(-1 \leq \nu \leq \frac{1}{2})$ of Poisson ratio ν.